ON REPRESENTATIONS OF POSITIVE INTEGERS BY $(a+c)^{\frac{1}{3}}x+(b+d)y$, $(a+c)x+\big(k(b+d)\big)^{\frac{1}{3}}y$, AND $\big(k(a+c)\big)^{\frac{1}{3}}x+l(b+d)y$

MOHAMED EL BACHRAOUI

Abstract. We use sums of Liouville type to count the number of ways a positive integer can be represented by the forms $(a+c)^{\frac{1}{3}}x+(b+d)y$, $(a+b)x+\big(k(c+d)\big)^{\frac{1}{3}}y$, and $\big(k(a+b)\big)^{\frac{1}{3}}x+l(c+d)y$ for nonnegative integers a,b,c,d,k,l,x,y under certain relative primality conditions.

1. Introduction

Throughout, let $\mathbb{N}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{Z}$, and $\mathbb{C}$ be the sets of positive integers, nonnegative integers, integers, and complex numbers respectively. The set of even positive integers and the set of odd positive integers will be written $2\mathbb{N}$ and $2\mathbb{N}-1$ respectively. We will use p to denote prime numbers and for simplicity we shall often write (m,n) rather than $\gcd(m,n)$. Next $\phi(n)$ will denote the Euler totient function and $\mu(n)$ will denote the Möbius mu function. For our current purposes we record the following well-known properties of these two functions. If $n>1$ and $s\in\mathbb{Z}\setminus\{0\}$, then

$$(1)\quad \phi(n)=n\prod_{p\mid n}\left(1-\frac{1}{p}\right),\ \sum_{d\mid n}\phi(d)=n,\ \sum_{d\mid n}\mu(d)=0,\ \sum_{d\mid n}\mu(d)d^s=\prod_{p\mid n}(1-p^s).$$

Further, it is easy to verify the following formula which is crucial in this paper. If $k,n\in\mathbb{N}$ such that $n>1$, then

$$(2)\qquad \sum_{\substack{1\le l<n\\ (l,n)=1}} l^k=\sum_{d\mid n}\mu(d)d^k\sum_{j=1}^{\frac{n}{d}}j^k=\sum_{d\mid n}\mu(d)d^k\sum_{j=1}^{\frac{n}{d}-1}j^k$$

$$=\sum_{d\mid n}\mu(d)\frac{d^k}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j\left(\frac{n}{d}\right)^{k+1-j},$$

where B_j denotes the jth Bernoulli number for which $B_1=-1/2$ and $B_{2j+1}=0$ for $j\in\mathbb{N}$ and the first few terms for even j are

$$B_0=1,\ B_2=1/6, B_4=-1/30, B_6=1/42, B_8=-1/30.$$

Definition 1. For $n\in\mathbb{N}$ let the sets $\mathcal{B}(n)$ and $\mathcal{B}'(n)$ be defined as follows:

$$\mathcal{B}(n)=\{(a,b,x,y)\in\mathbb{N}^4:\ ax+by=n\},$$

$$\mathcal{B}'(n)=\{(a,b,x,y)\in\mathbb{N}^4:\ ax+by=n,\ \text{and}\ \gcd(a,b)=\gcd(x,y)=1\}.$$

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For $n \in 2\mathbb{N}$ let the sets $\mathcal{O}(n)$ and $\mathcal{O}'(n)$ be defined as follows:

$$\mathcal{O}(n) = \{(a,b,x,y) \in (2\mathbb{N}-1)^4 : \ ax+by=n\},$$

$$\mathcal{O}'(n) = \{(a,b,x,y) \in (2\mathbb{N}-1)^4 : \ ax+by=n, \text{ and } \gcd(a,b)=\gcd(x,y)=1\}.$$

Definition 2. For $1 < n \in \mathbb{N}$ let the numbers $G'(n)$, $H'(n)$, and $I'(n)$ be defined as follows:

$$G'(n) = \#\{(a,b,c,d,x,y) \in \mathbb{N}_0^2 \times \mathbb{N}^4 : \ ((a+c)^{\frac{1}{3}}, (b+d), x, y) \in B'(n)\}$$

$$H'(n) = \#\{(a,b,c,d,k,x,y) \in \mathbb{N}_0^2 \times \mathbb{N}^5 : \ ((a+c), (k(b+d))^{\frac{1}{3}}, x, y) \in B'(n) \text{ and } (b,d)=1\}$$

$$I'(n) = \#\{(a,b,c,d,k,l,x,y) \in \mathbb{N}_0^2 \times \mathbb{N}^6 : \ ((k(a+c))^{\frac{1}{3}}, l(b+d), x, y) \in B'(n) \text{ and } (a,c)=(b,d)=1\}.$$

For $n \in 2\mathbb{N}$ let the numbers $J'(n)$, $K'(n)$, and $L'(n)$ be defined as follows:

$$J'(n) = \#\{(a,b,c,d,x,y) \in \mathbb{N}_0^2 \times \mathbb{N}^4 : \ ((a+c)^{\frac{1}{3}}, (b+d), x, y) \in \mathcal{O}'(n)\}$$

$$K'(n) = \#\{(a,b,c,d,k,x,y) \in \mathbb{N}_0^2 \times \mathbb{N}^5 : \ ((a+c), (k(b+d))^{\frac{1}{3}}, x, y) \in \mathcal{O}'(n) \text{ and } (b,d)=1\}$$

$$L'(n) = \#\{(a,b,c,d,k,l,x,y) \in \mathbb{N}_0^2 \times \mathbb{N}^6 : \ ((k(a+c))^{\frac{1}{3}}, l(b+d), x, y) \in \mathcal{O}'(n) \text{ and } (a,c)=(b,d)=1\}.$$

Williams in [7, Theorem 11.1] reproduced the Jacobi's four squares formula with the help of the following sum over the set $\mathcal{B}(n)$ which is due to Liouville [4]. For a positive integer n and an even function $f : \mathbb{Z} \to \mathbb{C}$ we have

$$\sum_{(a,b,x,y)\in\mathcal{B}(n)} \big(f(a-b) - f(a+b)\big) = f(0)\big(\sigma(n) - d(n)\big)$$

$$+ \sum_{\substack{d\in\mathbb{N} \\ d|n}} (1+2n/d-d) f(d) - 2 \sum_{\substack{d\in\mathbb{N} \\ d|n}} \big(\sum_{l=1}^{d} f(l)\big),$$

where $d(n)$ is the number of positive divisors of n and $\sigma(n)$ is the sum of these divisors. For integer representations which are obtained using this type of sums we refer to [5, 6]. In this note we shall give exact formulas for each of the numbers $G'(n)$, $H'(n)$, $I'(n)$, $J'(n)$, $K'(n)$, and $L'(n)$. Our main tool is the following theorem of the author on sums over the sets $\mathcal{B}'(n)$ and $\mathcal{O}'(n)$.

Theorem 1. [2] *(a) If $1 < n \in \mathbb{N}$ and $f : \mathbb{Z} \to \mathbb{C}$ is an even function, then*

$$\sum_{(a,b,x,y)\in\mathcal{B}'(n)} \big(f(a-b) - f(a+b)\big) = \big(f(0) + 2f(1) - f(n)\big)\phi(n) - 2 \sum_{\substack{1\le l<n \\ (l,n)=1}} f(l).$$

(b) If $n \in 2\mathbb{N}$ and $f : \mathbb{Z} \to \mathbb{C}$ is an even function, then

$$\sum_{(a,b,x,y)\in\mathcal{O}'(n)} \big(f(a-b) - f(a+b)\big) = \big(f(0) - f(n)\big)\phi(n).$$

2. The results

We will need the following lemma which is the analogue of Theorem 12.3 in [7].

Lemma 3. *(a) If $k, n \in \mathbb{N}$ such that $n > 1$, then*

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \left(\sum_{(a,b,x,y)\in\mathcal{B}'(n)} a^{2k-2s-1} b^{2s+1} \right)$$

$$= \frac{n^{2k}-2}{2}\phi(n) + \frac{1}{2k+1}\sum_{d|n}\mu(d)d^{2k}\left(\sum_{j=0}^{2k}\binom{2k+1}{j}B_j\left(\frac{n}{d}\right)^{2k+1-j}\right).$$

(b) If $k \in \mathbb{N}$ and $n \in 2\mathbb{N}$, then

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\left(\sum_{(a,b,x,y)\in\mathcal{O}'(n)} a^{2k-2s-1}b^{2s+1}\right) = n^{2k}\phi(n).$$

Proof. (a) Application of Theorem 1(a) to the even function $f(x) = x^{2k}$ yields

$$\sum_{(a,b,x,y)\in\mathcal{B}'(n)}\left((a-b)^{2k}-(a+b)^{2k}\right) = (2-n^{2k})\phi(n) - 2\sum_{\substack{1\le l<n\\(l,n)=1}} l^{2k}. \tag{3}$$

The left hand side of (3) simplifies to

$$-2\sum_{(a,b,x,y)\in\mathcal{B}'(n)}\sum_{s=0}^{k-1}\binom{2k}{2s+1}a^{2k-2s-1}b^{2s+1}$$
$$= -2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{(a,b,x,y)\in\mathcal{B}'(n)} a^{2k-2s-1}b^{2s+1}.$$

On the other hand, by virtue of identity (2) the right hand side of (3) becomes

$$(2-n^{2k})\phi(n) - 2\sum_{d|n}\mu(d)\frac{d^{2k}}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j(n/d)^{2k+1-j}.$$

Equating the left and the right hand sides of (3) and dividing by -2 yields

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{(a,b,x,y)\in\mathcal{B}'(n)} a^{2k-2s-1}b^{2s+1}$$
$$= \frac{(n^{2k}-2)}{2}\phi(n) + \sum_{d|n}\mu(d)\frac{d^{2k}}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j(n/d)^{2k+1-j},$$

as desired.
(b) Similar to the previous part with an application of Theorem 1(b) to the even function $f(x) = x^{2k}$. $\square$

Theorem 2. *(a) If $1 < n \in \mathbb{N}$, then*

$$G'(n) = H'(n) = I'(n) = \frac{7n^5-10n}{80}\prod_{p|n}\left(1-\frac{1}{p}\right) + \frac{n^3}{24}\prod_{p|n}(1-p) - \frac{n}{240}\prod_{p|n}(1-p^3).$$

(b) If $n \in 2\mathbb{N}$, then

$$J'(n) = K'(n) = L'(n) = \frac{n^5}{8}\prod_{p|n}\left(1-\frac{1}{p}\right).$$

Proof. (a) We have

$$\begin{aligned}
G'(n) &= \sum_{\substack{(a,b,c,d,x,y)\in\mathbb{N}_0{}^2\times\mathbb{N}^4\\ (a+c)^{1/3}x+(b+d)y=n\\ ((a+c)^{1/3},b+d)=(x,y)=1}} 1 \\
&= \sum_{(u,v,x,y)\in\mathcal{B}'(n)} \Big(\sum_{\substack{(a,c)\in\mathbb{N}_0\times\mathbb{N}\\ a+c=u^3}} 1\Big)\Big(\sum_{\substack{(b,d)\in\mathbb{N}_0\times\mathbb{N}\\ b+d=v}} 1\Big) \\
&= \sum_{(u,v,x,y)\in\mathcal{B}'(n)} u^3 v, \\
H'(n) &= \sum_{\substack{(a,b,c,d,k,x,y)\in\mathbb{N}_0^2\times\mathbb{N}^5\\ (a+c)x+(k(b+d))^{1/3}y=n\\ \big(a+c,(k(b+d))^{1/3}\big)=(x,y)=(b,d)=1}} 1 \\
&= \sum_{(u,v,x,y)\in\mathcal{B}'(n)} \Big(\sum_{\substack{(a,c)\in\mathbb{N}_0\times\mathbb{N}\\ a+c=u}} 1\Big)\Big(\sum_{e\mid v^3}\sum_{\substack{(b,d)\in\mathbb{N}_0\times\mathbb{N}\\ b+d=e\\ (b,d)=1}} 1\Big) \\
&= \sum_{(u,v,x,y)\in\mathcal{B}'(n)} u\sum_{e\mid v^3}\phi(e) = \sum_{(u,v,x,y)\in\mathcal{B}'(n)} uv^3,
\end{aligned}$$

and

$$\begin{aligned}
I'(n) &= \sum_{\substack{(a,b,c,d,k,l,x,y)\in\mathbb{N}_0^2\times\mathbb{N}^6\\ (k(a+c))^{1/3}x+l(b+d)y=n\\ \big(k(a+c),l(b+d)\big)=(x,y)=(a,c)=(b,d)=1}} 1 \\
&= \sum_{(u,v,x,y)\in\mathcal{B}'(n)} \Big(\sum_{e\mid u^3}\sum_{\substack{(a,c)\in\mathbb{N}_0\times\mathbb{N}\\ a+c=e\\ (a,c)=1}} 1\Big)\Big(\sum_{f\mid v}\sum_{\substack{(b,d)\in\mathbb{N}_0\times\mathbb{N}\\ b+d=f\\ (b,d)=1}} 1\Big) \\
&= \sum_{(u,v,x,y)\in\mathcal{B}'(n)} \sum_{e\mid u^3}\phi(e)\sum_{f\mid v}\phi(f) = \sum_{(u,v,x,y)\in\mathcal{B}'(n)} u^3 v.
\end{aligned}$$

This shows that $G'(n)=H'(n)=I'(n)=\sum_{(u,v,x,y)\in\mathcal{B}'(n)} u^3v$ and so we will be done if we prove that

$$\sum_{(a,b,x,y)\in\mathcal{B}'(n)} a^3 b = \frac{7n^5-10n}{80}\prod_{p\mid n}\left(1-\frac{1}{p}\right)+\frac{n^3}{24}\prod_{p\mid n}(1-p)-\frac{n}{240}\prod_{p\mid n}(1-p^3).$$

Taking $k=2$ in Lemma 3(a) gives

$$\begin{aligned}
&4\sum_{(a,b,x,y)\in\mathcal{B}'(n)} a^3b + 4\sum_{(a,b,x,y)\in\mathcal{B}'(n)} ab^3 \\
&= \frac{n^4-2}{2}\phi(n)+\frac{1}{5}\sum_{d\mid n}\mu(d)d^4\left(B_0\frac{n^5}{d^5}+5B_1\frac{n^4}{d^4}+10B_2\frac{n^3}{d^3}+10B_3\frac{n^2}{d^2}+5B_4\frac{n}{d}\right),
\end{aligned}$$

or equivalently,

$$8\sum_{(a,b,x,y)\in\mathcal{B}'(n)} a^3b$$

$$= \frac{n^4-2}{2}\phi(n) + \frac{n^5}{5}\sum_{d|n}\mu(d)d^{-1} + \frac{n^3}{3}\sum_{d|n}\mu(d)d - \frac{n}{30}\sum_{d|n}\mu(d)d^3.$$

With the help of the properties (1) the previous identity yields

$$\sum_{(a,b,x,y)\in\mathcal{B}'(n)} a^3b$$

$$= \frac{n^5-2n}{16}\prod_{p|n}\left(1-\frac{1}{p}\right) + \frac{n^5}{40}\prod_{p|n}\left(1-\frac{1}{p}\right) + \frac{n^3}{24}\prod_{p|n}(1-p) - \frac{n}{240}\prod_{p|n}(1-p^3)$$

$$= \frac{7n^5-10n}{80}\prod_{p|n}\left(1-\frac{1}{p}\right) + \frac{n^3}{24}\prod_{p|n}(1-p) - \frac{n}{240}\prod_{p|n}(1-p^3).$$

This completes the proof of part (a). Part (b) follows similarly and the details are left to the reader. □

3. Final remarks

With the help of a result of Williams in [7] we shall give in this section formulas for the following related numbers:

$$G(n) = \#\{(a,b,c,d,x,y)\in\mathbb{N}_0^2\times\mathbb{N}^4 : \ ((a+c)^{\frac{1}{3}},(b+d),x,y)\in\mathcal{B}(n)\}$$

$$H(n) = \#\{(a,b,c,d,k,x,y)\in\mathbb{N}_0^2\times\mathbb{N}^5 : \ ((a+c),(k(b+d))^{\frac{1}{3}},x,y)\in\mathcal{B}(n) \text{ and } (b,d)=1\}$$

$$I(n) = \#\{(a,b,c,d,k,l,x,y)\in\mathbb{N}_0^2\times\mathbb{N}^6 : \ ((k(a+c))^{\frac{1}{3}},l(b+d),x,y)\in\mathcal{B}(n) \text{ and } (a,c)=(b,d)=1\}.$$

For this purpose, let $\sigma_m(n)$ be the sum of the mth powers of the divisors of n where $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$.

Theorem 3. *If* $1<n\in\mathbb{N}$, *then*

$$G(n) = H(n) = I(n) = \frac{7}{80}\sigma_5(n) + \left(\frac{1}{24}-\frac{1}{8}n\right)\sigma_3(n) - \frac{1}{240}\sigma(n).$$

Proof. On the one hand, by an argument as in the proof of Theorem 2(a) we find

$$H(n) = G(n) = I(n) = \sum_{(u,v,x,y)\in\mathcal{B}(n)} u^3v.$$

On the other hand, it is easily seen that

$$\sum_{(u,v,x,y)\in\mathcal{B}(n)} u^3v = \sum_{m=1}^{n-1}\sigma_3(m)\sigma(n-m).$$

Moreover, by Williams [7, Example 12.3, p. 128] we have

$$\sum_{m=1}^{n-1}\sigma_3(m)\sigma(n-m) = \frac{7}{80}\sigma_5(n) + \left(\frac{1}{24}-\frac{1}{8}n\right)\sigma_3(n) - \frac{1}{240}\sigma(n).$$

Combining the previous identities yields the result. □

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Dept. Math. Sci, United Arab Emirates University, PO Box 17551, Al-Ain, UAE

E-mail address: melbachraoui@uaeu.ac.ae